# 1 Extending Cartesian Grid Methods

Cartesian grid methods for super-sonic compressible fluid flows are usually *shock-capturing* finite volume schemes.

## 1.1 Shock-capturing Methods

- Most efficient class of numerial methods for solutions with discontinuities. Simpler implementation (especially in 3D) than front tracking, considerable better approximation quality than particle methods.
- Upwinding by characteristic decomposition in all characteristic fields.
- Eulerian frame of reference, fixed grid is employed to approximate moving flow field.
- Correct approximation of discontinuities requires update formula in conservation form.
- Maximal order of accuracy most easily achieved on structured Cartesian grids. Observation: Very elaborated schemes (ENO, WENO, etc.) typically do not achieve the proposed order of accuracy on unstructured meshes.
- BUT: Structured grids are typically not flexible enough for some applications.

Natural idea: Extend the *discretization* of a Cartesian grid method to handle non-Cartesian problems properly.

Key question: *How to do this without loosing essential mathematical properties?*

## 1.2 Embedded Boundary Methods

- Representation of a moving *boundary*.
- A shock-capturing scheme is only used on the internal side of the boundary.
- Methods that diffuse the boundary in one cell:
  - Internal ghost cell values [7] can be set directly on grid data. Compare Fig. 1.
  - Numerical stencil is not modified at the boundary.
  - Usually not conservative, but flux correction / redistribution step after update possible [10, 8].
  - Often implemented with *implicit* geometry representation (level set equation), because sharp boundary representation not required [9].
- Methods that represent the boundary sharply (cut-cell techniques):
  - Exact boundary flux is considered. Cartesian stencil is modified.
  - Conservative by construction. Goal: Incorporate boundary flux without stability restrictions.
  - Merging of small and uncutted cells [11, 4].

  - Update small cells with full time-step and add waves algebraically to neighboring cells [3, 2].

- Usually implemented with *explicit* geometry representation (curves) to avoid numerical smearing. An efficient mapping into the Cartesian grid is required.

## 1.3 Ghost-Fluid Methods

- Representation of a moving *interface*.
- Interface can be an important discontinuity or a phase boundary.
- Interface is diffused across one cell. Usually not conservative across interface.
- Ghost cell values on *both* sides of the interface are used to treat both sides of the interface seperately. Can involve even *two* different finite volume schemes.
- Data on full grid is not necessarily meaningful for both schemes. Blanking of unused cells required!
- Interface is propagated typically based on Rankine-Hugoniot relations. *This ensures the approximation of the correct weak solution although the conservation property is violated across the interface.*
- Ghost cells overlap with internal cells for the other side! See Fig. 2.
- Cells previously used as ghost cells can become internal cell when interface is moving. A reinitialization of such cells seems appropriate, although some authors do not consider this step as necessary, cf. [6, 5, 1].
- Implicit geometry representation with level set: Advection with superimposed velocity field. Problems:

  1. Sufficient resolution to avoid excessive numerical smearing.
  2. Accurate approximation of interface velocity field in multi-dimensional problems.

- Explicit geometry representation with curves: front tracking for some discontinuities.

  - Shock-capturing method is still required, because not all discontinuities are tracked.
  - Difficult in multiple dimensions when fronts intersect.

## 1.4 Comparison and Conclusions

- The first three steps of a ghost-fluid method are similar to an embedded boundary method with internal ghost cell usage.
- A level set representation with numerical *advection step* is the essential component in implementing ghost-fluid methods in multiple space-dimensions effectively.
- *A framework for ghost-fluid methods can support embedded boundary methods with level set representation and internal ghost cell usage by the way!*

- Diffusive embedded boundary methods require only the signed distance. Approximated normal will usually be sufficiently accurate to construct diffused boundary information.
- A sharp boundary representation even as a correction step needs the exact normal, which therefore has to be calculated and stored additionally. This increases the storage requirements considerably and is presumably the main reason why the level set approach is typically *not* taken in methods with sharp boundary (note that the boundary is originally of lower dimension).
- Research idea:
  - Construction of a *level-set-based* framework with sharp boundary representation for embedded boundaries and ghost-fluid methods.
  - Avoid the time-step restriction for explicit schemes due to small cells by appropriate cell merging, but ensure the usage of the correct boundary flux as in an unstructured method.
  - The framework will be fully conservative and application of the right boundary flux will capture discontinuities that *interact* with the interface/boundary correctly. *This is not guaranteed in diffused embedded boundary methods and non-conservative ghost-fluid approaches.*
  - First-order accuracy along the boundary would be enough. Higher-order accuracy requires extremely difficult multi-dimensional consideration of boundary flux [2] and has not been demonstrated in 3D by now.
  - If the boundary normal is approximated from the level set function AMR can moderate the error.

# References

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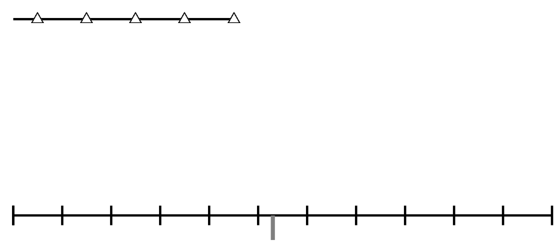
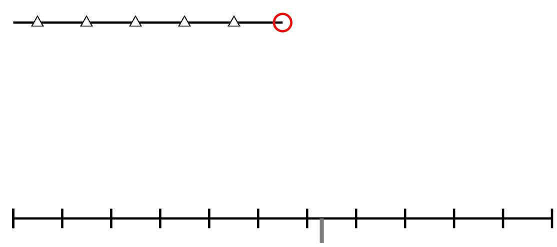
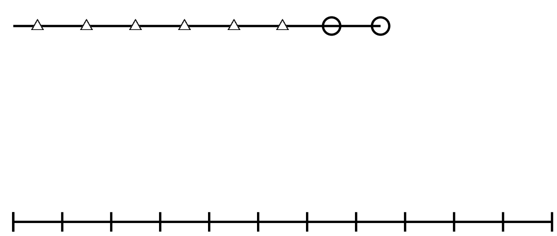
|---|---|
| Propagate boundary |  |
| 1. Calculate new volumes<br>2. Reinitialize cell with $V_i^n < \frac{1}{2}, V_i^{n+1} = 1$ |  |
| 1. Construct ghost cell values<br>2. Solve |  |

Figure 1: Principle sketch of an embedded boundary method that uses only internal ghost cells to incorporate the boundary.

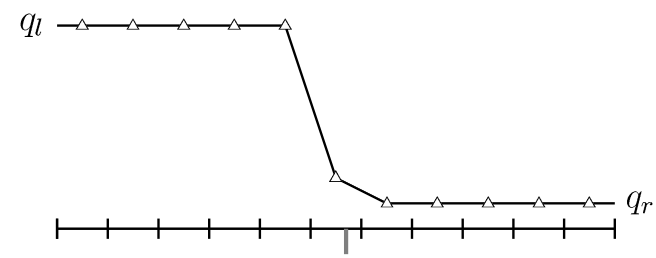
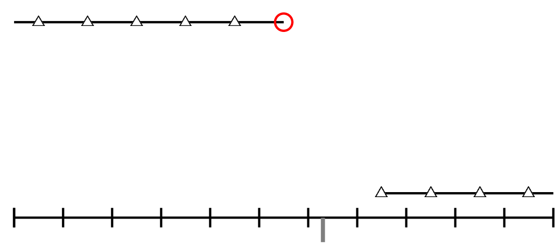
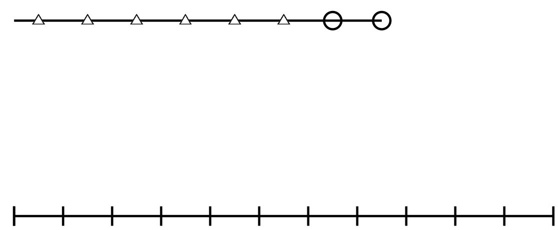
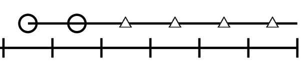
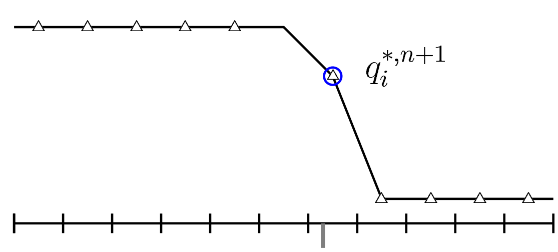
| | |
|---|---|
| Advect interface, e.g. $$d = \frac{f(q_l) - f(q_r)}{q_l - q_r}$$ |  |
| 1. Calculate new volumes<br>2. Reinitialize cell with $V_i^n < 1, V_i^{n+1} = 1$ |  |
| 1. Construct ghost fluid values for left side<br>2. Solve with scheme A |  |
| 1. Construct ghost fluid values for right side<br>2. Solve with scheme B |  |
| Construct new intermediate value $$q_i^{*,n+1} = V_i^{n+1} q_{i-1}^{n+1} + (1 - V_i^{n+1}) q_{i+1}^{n+1}$$ |  |

Figure 2: Principle sketch of a ghost-fluid method.

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